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Solution of a coupled creeping-flow problem by the Wiener–Hopf method

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Abstract. An analytical solution is presented of the problem of outflow of a viscous fluid from an infinite array of parallel semi-infinite ducts. After a local analysis of the singularities existing at the wall edges, the Wiener–Hopf method is applied, yielding an explicit quadrature expression of the solution. The coefficient of the singular term, which was left undetermined by the local analysis, is calculated exactly, and various relevant parameters of the flow are calculated by numerical Fourier transforms and plotted.

1. Introduction

The majority of the available solutions of the Stokes equations refer to inflow and/or outflow boundary conditions assigned at a given station. Typical is Joseph's [1–2] modal expansion of flow in a finite duct. However, it is difficult to think of a physical situation of Stokes flow in which two properties are known a priori at a given station. In practice, the entrance of a flow field is always the exit of some other flow field, and therefore the coupling of two infinite flow regions of different shape is an interesting problem.

In these coupled creeping-flow problems, generally, a singularity arises at the transition point between different kinds of boundaries. This singularity makes it difficult to attach them by standard techniques. Coupled modal expansions could be used when both regions to be connected admit one, but the resulting infinite set of linear equations does not provide a good approximation when it is truncated because the presence of the singularity makes the mode series slowly converging. The presence of a singularity is also an obstacle to the straightforward application of numerical methods.

An example of coupled creeping flow problems is given in [3], whose authors solved, by a Green function technique, the problem of the flow generated by a pointlike fluid injection through the walls of an infinite plane duct, and also in [4], where a solution is given for flow through a slotted wall separating two semi-infinite half-spaces.

The Wiener–Hopf factorization method [5] is a general technique to solve linear problems characterized by a boundary condition being imposed on some quantity along a half of a straight line and on a different quantity along the other half. Its perhaps most well known applications belong to the field of electromagnetic-wave diffraction theory, but the method has also been used, already for about 30 years, in the theory of elasticity and in fluid mechanics. Within the field of coupled creeping-flow problems, two geometries amenable to application of the Wiener–Hopf theory are those of the merging of two streams arriving along the opposite sides of an infinite half-plane wall (Fig. 1a) and of flow out of an infinite array of parallel semi-infinite plane ducts (Fig. 1b). The first, however, has a well-known analytical solution by other means, as will be recalled in the next section; therefore we shall mainly concentrate on the second, which we shall call the periodic-outlet problem.

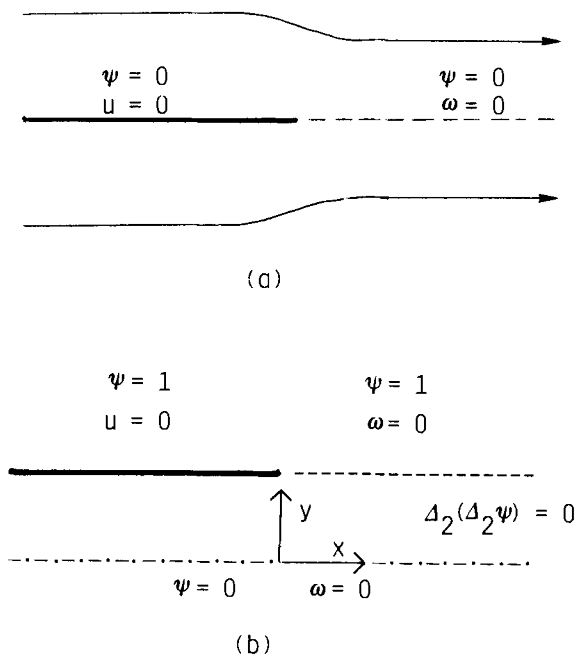

Fig. 1. Flow problems amenable to the Wiener–Hopf technique: (a) merging of two streams arriving on opposite sides of a semi-infinite plane wall; (b) outflow from an infinite array of parallel plane ducts.

The plan of the paper is as follows: in Sec. 2 we shall briefly recall the separable solutions of Stokes' equations that describe the local singular behaviour near corners in the boundary of creeping flows, and which also happen to give the exact solution of flow about an infinite half-plane; in Sec. 3 we shall describe the application of the Wiener–Hopf method to the periodic-outlet problem and in Secs. 4 and 5 we shall show the analytical and numerical results yielded by this approach.

2. Analysis of wedge singularities

Moffat [6] studied viscous eddies near sharp corners, pointing out that a local solution can be found in the class of separable solutions of the Stokes equations in which the streamfunction ψ takes the form $\psi = r^{\lambda} f(\theta)$ (with r and θ polar coordinates). Such an analysis can be extended to all the singularities most frequently encountered in coupled flow problems.

In particular, in the periodic-outlet problem in Fig. 1b the boundary conditions before the exit (on the duct wall) require $\psi = \partial\psi/\partial y = 0$ and after the exit (for symmetry reasons) $\psi = \partial^2\psi/\partial y^2 = 0$. The local solution is obtained by applying these boundary conditions on the edge of a half plane (that is, by pushing the opposite walls to infinity); in other words, by solving the problem of the merging of two streams arriving along the opposite sides of a half-plane wall.

Inserting the expression $\psi = r^{\lambda} f(\theta)$ into the Stokes equations, in the streamfunction-vorticity form,

$$\Delta_2 \psi \overset{\text{def}}{=} \omega \; ; \qquad \Delta_2 \omega = 0 \, , \quad \left(\text{with } \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{1}$$

shows that equations (1) are satisfied provided $f(\theta)$ is a solution of the ordinary differential equation

$$\left[\frac{d^2}{d\theta^2} + \lambda^2\right]\left[\frac{d^2}{d\theta^2} + (\lambda - 2)^2\right] f = 0 . \tag{2}$$

The boundary conditions relevant to the problem at hand generate a homogeneous boundary-value problem for $f(\theta)$ in which λ plays the role of an eigenvalue. This eigenvalue problem has an infinite number of eigensolutions, corresponding to a discrete sequence of values of λ, a superposition of which is able to represent the solution of (1) that satisfies general boundary conditions at infinity (or, to be more precise, at an outer radius $r = R$ which may be arbitrarily large). The lowest eigenvalue λ yields the leading term of such an expansion near $r = 0$, and is 3/2. The corresponding eigensolution is given by

$$\psi = cr^{3/2}[\sin(3\theta/2) + \sin(\theta/2)] , \tag{3}$$

where the value of the multiplicative constant c is left undetermined by local analysis, and will be calculated later.

Similar expressions of the form $\psi = r^{\lambda} f(\theta)$, each with its own characteristic value of λ, may be found which describe the local singular behaviour of the solution of the Stokes equations near wedges of any angle and with various kinds of boundary conditions assigned on the two sides. Each case only requires the solution of an eigenvalue problem for the linear constant-coefficient ordinary differential equation (2) with the boundary conditions appropriate to each case imposed at the corresponding angle θ.

3. The Wiener–Hopf method

The Wiener–Hopf method [5] is a general technique to solve linear problems characterized by giving a boundary condition on some quantity along a half of a straight line and on a different quantity along the other half. The periodic-outlet problem (Fig. 1b) can be recognized to belong to this class. As is shown in the figure, the Stokes equations (1) must be solved with conditions specifying that $\psi = 0$ and $\omega = 0$ on the axis ($y = 0$) and $\psi = 1$ at the wall ($y = 1$) for $-\infty < x < \infty$, and $\psi_y = 0$ and $\omega = 0$ at the wall for $-\infty < x < 0$ and $0 < x < \infty$, respectively.

If all boundary conditions were given for $-\infty < x < \infty$, such a problem would be solvable by Fourier transformation. Suppose, for instance, that the value of $\omega_1(x) \stackrel{\text{def}}{=} \omega(x, 1)$ were assigned for $-\infty < x < \infty$. Fourier-transforming all quantities with respect to x and denoting transforms by capital letters, as for instance in

$$\Omega_1(k) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \omega_1(x)\, e^{-ikx}\, dx , \tag{4}$$

gives the ordinary-differential-equation problem

$$\Psi_{yy} - k^2\Psi = \Omega ; \qquad \Omega_{yy} - k^2\Omega = 0 , \tag{5}$$

with $\Psi(k, 0) = \Omega(k, 0) = 0$, $\Psi(k, 1) = 2\pi\delta(k)$, $\Omega(k, 1) = \Omega_1(k)$. The solution of (5) is immediate and yields

$$\Psi(k, y) = \frac{y \sinh k \cosh ky - \cosh k \sinh ky}{2k \sinh^2 k} \Omega_1(k) + 2\pi y \delta(k) . \tag{6}$$

In particular, the Fourier transform of $u_1(x) \overset{\text{def}}{=} u(x, 1) = \psi_y(x, 1)$ turns out to be given by

$$U_1(k) = K(k)\Omega_1(k) + F(k) , \tag{7}$$

where

$$K(k) = \frac{\sinh k \cosh k - k}{2k \sinh^2 k} ; \qquad F(k) = 2\pi\delta(k) . \tag{8}$$

If it were now possible to determine two functions $\Omega_1(k)$ and $U_1(k)$ satisfying (7) and such that $\Omega_1(k)$ is the transform of a function that is zero for $x > 0$ and $U_1(k)$ is the transform of a function that is zero for $x < 0$, the original problem would be solved. The determination of two such functions constitutes a Wiener–Hopf problem.

The starting point for the solution of a Wiener–Hopf problem is the observation that the Fourier transform of a function that is zero for $x > 0$, such as $\Omega_1(k)$, is an analytic function in the complex half-plane $\text{Im}(k) > 0$, as follows directly from its definition (4), and conversely the Fourier transform of a function that is zero for $x < 0$, such as $U_1(k)$, is analytic for $\text{Im}(k) < 0$. Therefore the problem can be restated within the theory of analytic functions of a complex variable.

According to the standard procedure due to Hilbert, to solve this problem we must first factorize the kernel $K(k)$ into one factor, say $K^+(k)$, which has to be analytic and without zeros for $\text{Im}(k) > 0$, and a second factor $K^-(k)$ analytical and without zeros for $\text{Im}(k) < 0$ (we postpone showing how this factorization may be attained). Dividing both sides of (7) by $K^-(k)$ then gives

$$U_1(k)/K^-(k) = K^+(k)\Omega_1(k) + F(k)/K^-(k) .$$

Now, if we can split the known term $\bar{F}(k) = F(k)/K^-(k)$ into the sum of a function $F^+(k)$ analytic for $\text{Im}(k) > 0$ and a function $F^-(k)$ analytic for $\text{Im}(k) < 0$, we can immediately write down the solution of the Wiener–Hopf problem (7) as

$$U_1(k) = K^-(k)F^-(k) ; \qquad \Omega_1(k) = -F^+(k)/K^+(k) . \tag{9}$$

In order to perform the above procedure in practice, we must be able to split a function $\bar{F}(k)$ into the sum of two parts $F^+(k)$ and $F^-(k)$, one analytic in the half-plane $\text{Im}(k) > 0$ and the other in the half-plane $\text{Im}(k) < 0$. Such a splitting is furnished by the well-known formulae due to Cauchy:

$$F^+(k) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{F}(k')}{k - k' + i0} \, dk' , \tag{10a}$$

$$F^-(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\bar{F}(k')}{k - k' - i0} \, dk' , \tag{10b}$$

where the notation $+\mathrm{i}0$ $(-\mathrm{i}0)$ indicates that the path of integration must run below (above) the pole $k' = k$, and equations (10) are valid for real k and, each, in the half-plane where the corresponding function is analytic, values in the other half-plane having to be obtained by analytic continuation. The factorization of the function $K(k)$, required in Hilbert's procedure, can be simply obtained by applying the splitting formulae (10) to $\log K$.

4. Application to the periodic-outlet problem

For the periodic-outlet problem, in which $K(k)$ and $F(k)$ are given by (8), the splitting of $\bar{F}(k)$ is immediate: by virtue of the properties of the δ-function, $\bar{F}(k) = 2\pi\delta(k)/K^-(k) = 2\pi\delta(k)/K^-(0)$, and inserting this result into (10) gives:

$$F^+(k) = -\frac{1}{\mathrm{i}K^-(0)}\frac{1}{k+\mathrm{i}0}, \qquad F^-(k) = \frac{1}{\mathrm{i}K^-(0)}\frac{1}{k-\mathrm{i}0}. \tag{11}$$

The factorization of $K(k)$, however, cannot be obtained analytically, because the integrals obtained after putting $\log K$ in the place of $\bar{F}(k)$ in (10) cannot be expressed in closed form. Nevertheless, it is possible to obtain a few useful limiting values. In particular, the fact that $K(k)$ is real for k real implies that the complex poles and zeros of $K(k)$ come in conjugate pairs. It follows that $K^+(k)$ and $K^-(k)$, each of which admits all and only the poles and zeros of $K(k)$ characterized by $\mathrm{Im}(k) < 0$ and $\mathrm{Im}(k) > 0$ respectively, have the symmetry property that $K^+(k) = [K^-(k^*)]^*$. In addition, since $K(k)$ is an even real function, K^+ and K^- must have even real and odd imaginary part on the real k axis. Therefore, from the value $K(0) = 1/3$ it may immediately be deduced that $K^+(0) = K^-(0) = 1/\sqrt{3}$, and from the behaviour at infinity, $K(k) \simeq 1/2|k|$, that $K^+(k) \simeq (-2\mathrm{i}k)^{-1/2}$ and $K^-(k) \simeq (2\mathrm{i}k)^{-1/2}$, where the square root with positive real part is understood.

The above information is sufficient to determine the coefficient c of the singular term in (3) exactly. In fact, it may be obtained from (9) that, for $k \to \infty$,

$$U_1(k) \simeq \frac{\sqrt{3}}{2}(-\mathrm{i}k)^{-3/2}; \qquad \Omega_1(k) \simeq (6/\mathrm{i}k)^{1/2}. \tag{12}$$

The theorems that relate the behaviour at infinity of transforms with the behaviour near zero of the transformed functions then show that, for $x \to 0$,

$$u_1(x) \simeq \left(\frac{6x}{\pi}\right)^{1/2}; \qquad \omega_1(x) \simeq \left(\frac{6}{-\pi x}\right)^{1/2}. \tag{13}$$

By comparison with the expression of either $u_1(x)$ or $\omega_1(x)$ obtained from (3) the coefficient c is thus found to equal $\sqrt{3}/\pi$.

5. Numerical factorization of analytic functions

The Wiener–Hopf method is generally used only as a purely analytical solution procedure. The fact that the factorization of $K(k)$ is not expressed in closed form, however, does not mean that the method cannot or should not be applied: the solution can still be expressed in a quadrature form, that is, a form that contains some integrals, and these integrals can be calculated numerically. The result will still be a direct solution, that is, one obtained through a noniterative procedure, and will be obtained with greater speed and, if some care is applied, greater precision than iterative numerical solutions.

A key observation for the numerical factorization of analytic functions is that, although we are dealing with functions of a complex variable, it is sufficient for all purposes to calculate integrals of the form (10) on the real axis only.

In particular, we are interested in computing (10) with

$$\log K = \log\left(\frac{\sinh k \cosh k - k}{2k \sinh^2 k}\right) \tag{14}$$

in place of $\bar{F}$. Such integrals (actually, only one of them need be computed because $K^+(k) = [K^-(k^*)]^*$) are conveniently expressed in terms of numerical Fourier transforms: they can be computed by Fourier-transforming $\bar{F}(k)$ (for instance, by a Fast Fourier Transform), multiplying the result by the transform of the convolution kernel $[2\pi \mathrm{i}(k \pm \mathrm{i}0)]^{-1}$ (which is a step function) and then reverse-transforming.

There is, however, a preliminary step that must be taken before the above procedure can be applied. The function defined by (14) is infinite at infinity and thus non-integrable in the ordinary sense. Analytically, this difficulty is solved by calculating the integral for non-real k first and then analytically prolongating it back to the real axis or by suitably extending the definition of integral, for instance by introducing the Cauchy principal value, which turns out to be appropriate in the context of Fourier transforms.

From the numerical point of view it is preferable to preliminarily divide K by an analytically factorable function which eliminates the offending behaviour at infinity (such a function is sometimes introduced, under the name of "neutralizer", during the proof of analytical theorems in order to show that a result initially obtained for smooth functions continues to hold in the presence of singularities). The easiest way to obtain an analytically

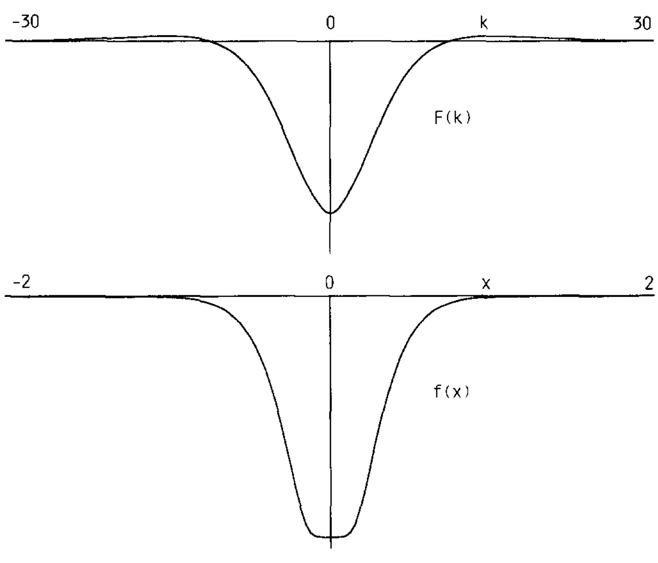

Fig. 2. The function $F(k) = \log(K/|N^+|^2)$ and its numerical Fourier transform $f(x)$.

factorable function is to actually construct it as the product of two factors with the required analyticity properties. What we need is a function that, just as, say, K^+, has no singularities on the real axis nor in the upper half-plane and behaves at infinity as $(-2ik)^{-1/2}$; moreover we should like this behaviour to be approached exponentially, as is true with regard to $K(k)$. A suitable function is $N^+(k) = \tanh[(-2ik)^{1/2}/2]/(-2ik)^{1/2}$. Therefore, our final numerical procedure will be to insert the function $\bar{F}(k) = \log(K/N^+N^-) = \log(K/|N^+|^2)$ into the integral (10a) and then, after the numerical computations have been performed, re-multiply the result by the neutralizer $N^+(k)$. The resulting function $\bar{F}(k)$ and its numerical Fourier transform are shown in Fig. 2.

Once we have K^+ (and thus, automatically, K^-) in numerical form, equations (12) immediately give $\Omega_1(k)$ and $U_1(k)$, from which, by two more numerical Fourier transforms, we can get $\omega_1(x)$ and $u_1(x)$. The result, again calculated by accounting for the singularities analytically in order to avoid excessive discretization errors, is shown in Figs 3 and 4.

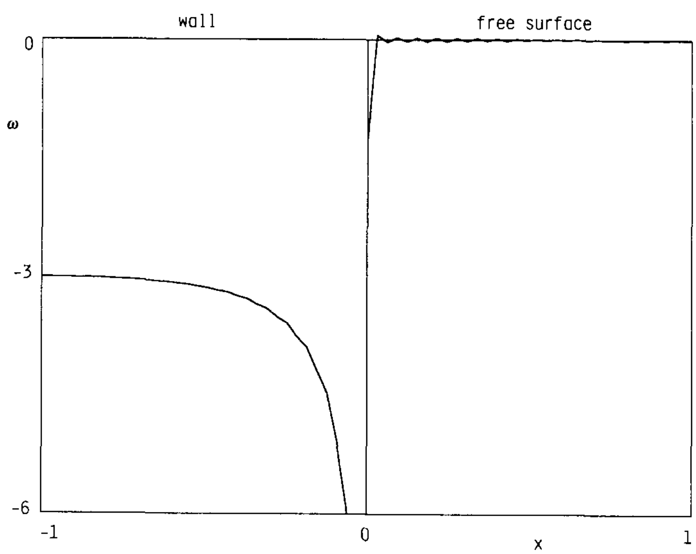

Fig. 3. Skin friction (vorticity) at the duct wall calculated after the numerical factorization of the kernel $K(k)$.

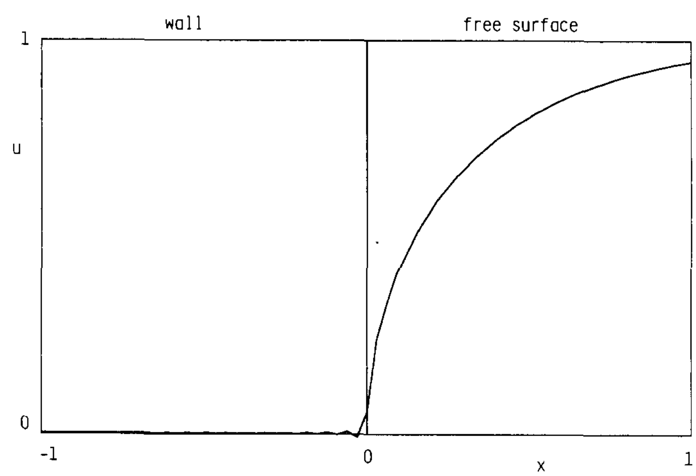

Fig. 4. Fluid velocity on the continuation of the wall into the free flow region.

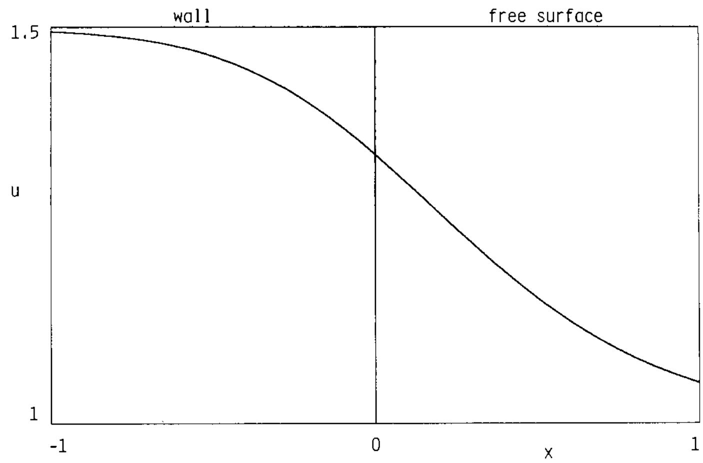

Fig. 5. Fluid velocity on the symmetry plane between two successive duct walls, both inside and outside the ducts.

Finally, to give an example of how we can also calculate, from (6), any other interesting quantity, Fig. 5 shows the velocity profile $u(x, 0)$ on the axis.

6. Conclusions

The Wiener–Hopf method has enabled us to obtain an explicit quadrature solution of the problem of outflow of a viscous fluid from an infinite array of parallel plane ducts. This solution can be exploited analytically to determine parameters of the motion, such as the coefficient of the asymptotic singular expression of the streamfunction near the edge of the plane wall, or numerically to calculate any desired quantity by explicit integration. To this end it has been shown that the integrals of complex analytic functions that characterize the Wiener–Hopf method, once recast as integrals on the real axis, can be efficiently computed by numerical Fourier transforms, provided a neutralizer is introduced to take care of singularities. Plots of a few interesting quantities have been given.

In conclusion, a new analytical solution of a coupled creeping-flow problem has been demonstrated. Besides the obvious possibility of using this solution for the test of numerical codes, it should be emphasized that the FFT-based calculation method adopted is very fast, and can be used as an effective component of numerical algorithms for the iterative solution of more complicated problems where the solution of a, possibly inhomogeneous, Stokes problem appears as an intermediate step in the iteration.

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